ELSEVIER

Journal of Geometry and Physics 39 (2001) 1–8

JOURNAL OF GEOMETRY AND PHYSICS

Colored brackets and 2-manifolds

David A. Richter*

Mathematics Department, Southeast Missouri State University, Cape Girardeau, MO 63701, USA

Received 11 April 2000

Abstract

Frequently, although a set of matrix differential operators will not be closed under, say, a Lie bracket, it might be closed under more general Lie Γ-graded brackets. Some of these operator products are defined using *commutation factors* for Γ. The classification of commutation factors, and hence Lie Γ-graded brackets, is one of the stepping stones in an attempt to classify algebras of matrix differential operators, generalizing a question outline in [Am. J. Math. 114 (6) (1992) 1163]. Many examples of Lie *super*algebras of matrix differential operators exist, but there is still a question of what other possible algebra products may close a space of matrix differential operators. Studying Lie color algebras and Lie color superalgebras represents an attempt in this direction. It is a curious fact that commutation factors for groups of the form $(\mathbb{Z}/2\mathbb{Z})^n$ coincide with homeomorphism-equivalence classes of connected compact 2-manifolds, so this coincidence is also given in Section 4 of this paper.

MSC: 17B75; 57Nxx

Subj. Class.: Lie groups; Lie algebras

Keywords: Lie color algebras; Lie color superalgebras; Commutation factors; 2-manifolds

1. Introduction

This note contains a complete classification of commutation factors for groups of the form $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^n$. In his classification of Lie superalgebras, [1], the author asks the question of classifying the "generalized" superalgebras, those graded by a group of the form $(\mathbb{Z}/2\mathbb{Z})^n$, instead of merely $\mathbb{Z}/2\mathbb{Z}$. However, before one can answer this question, one must first determine what possible Lie Γ-graded structures might exist, and to answer this one must classify commutation factors for Γ. The authors of [3] obtain this classification for $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^n$, but using slightly different terminology and notation. This note aims

* Tel.: +1-573-651-2773.
E-mail address: drichter@semovm.semo.edu (D.A. Richter).

0393-0440/01/$ – see front matter
PII: S0393-0440(00)00063-2

to clarify this classification in the terms of [4] and to relate this to the classification of connected compact 2-manifolds. As it turns out, once one defines a particular equivalence relation on the set of commutation factors for $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^n$, the equivalence classes, also referred in this paper to as *reduced* commutation factors, correspond exactly with homeomorphism-equivalence classes of connected compact 2-manifolds.

The terminology "color" comes from [3]. Since every commutation factor gives rise to a Lie Γ-graded bracket, sometimes we refer to commutation factors as *colored brackets*.

2. Generalities

First we need to discuss the general framework around commutation factors. These are some results which follow very easily from the definitions.

Definition 2.1. Let Γ be an abelian group. A commutation factor is a map $\epsilon : \Gamma \times \Gamma \to \mathbb{C}^+$ satisfying

$$\begin{aligned}
&\epsilon(a, b)\epsilon(b, a) = 1,\\
&\epsilon(a_1 + a_2, b) = \epsilon(a_1, b)\epsilon(a_2, b),\\
&\epsilon(a, b_1 + b_2) = \epsilon(a, b_1)\epsilon(a, b_2)
\end{aligned}$$

for all $a, a_1, a_2, b, b_1, b_2 \in \Gamma$. The diagonal map $\delta : \Gamma \to \mathbb{C}^+$ associated with ϵ is the map

$$\delta : a \mapsto \epsilon(a, a).$$

Proposition 2.2. *Let Γ be an abelian group and let ϵ be a commutation factor. Then* (*a*) $\epsilon(g, 0) = \epsilon(0, g) = 1$ *for all* $g \in \Gamma$, (*b*) *the diagonal map δ is a homomorphism, and* (*c*) *the image of δ lies in the two-element group* $C_2 = \{\pm 1\}$.

Proof. (a) Notice that the map $a \mapsto \epsilon(a, g)$, by definition, is a homomorphism. Thus, we have

$$\epsilon(0, g) = \epsilon(0 + 0, g) = \epsilon(0, g)\epsilon(0, g).$$

(b) Since $\epsilon(0, g) \neq 0$, we may divide to obtain $\epsilon(0, g) = 1$.

$$\begin{aligned}
\delta(a + b) &= \epsilon(a + b, a + b) = \epsilon(a, a + b)\epsilon(b, a + b)\\
&= \epsilon(a, a)\epsilon(a, b)\epsilon(b, a)\epsilon(b, b) = \epsilon(a, a)\epsilon(b, b) = \delta(a)\delta(b).
\end{aligned}$$

(c) Follows from (b) and the fact that $\delta(a)^2 = 1$. □

As is the case with the category of Lie algebras, there is a universal object which plays a rôle similar to that of $\mathfrak{gl}_n\mathbb{C}$ for Lie algebras. That is, there is a "general linear" Γ-graded algebra for every commutation factor ϵ. We construct this as follows. Suppose ϵ is a commutation factor for Γ and let V be a Γ-graded vector space,

$$V = \bigoplus_{a\in\Gamma} V_a.$$

The general linear Γ-graded algebra for ϵ is the space

$$\mathfrak{gl}(V,\epsilon) = \bigoplus_{a\in\Gamma} \mathfrak{gl}(V,\epsilon)_a,$$

where for each $a \in \Gamma$,

$$\mathfrak{gl}(V,\epsilon)_a = \bigoplus_{b\in\Gamma} \mathrm{Hom}(V_b, V_{a+b})$$

equipped with the bracket

$$\langle u, v\rangle = u \circ v - \epsilon(a,b) v \circ u$$

for $u \in \mathfrak{gl}(V,\epsilon)_a$ and $v \in \mathfrak{gl}(V,\epsilon)_b$.

In this paper, we do not wish to study Lie Γ-graded algebras in much detail, but, due to a generalization of Ado's theorem, [4], we may give the following definition of Lie Γ-algebras.

Definition 2.3. Suppose ϵ is a commutation factor for Γ. A space $\mathfrak{g}$ equipped with a bracket is a Lie Γ-graded algebra for ϵ if there is a vector space V and a map

$$\mathfrak{g} \to \mathfrak{gl}(V,\epsilon).$$

If $\mathfrak{g} = \oplus_{a\in\Gamma}\, \mathfrak{g}_a$ is a Lie Γ-graded algebra associated with ϵ, then the expected identities hold, Γ-graded skew-symmetry

$$\langle u, v\rangle + \epsilon(a,b)\langle v, u\rangle = 0, \tag{1}$$

and the Γ-graded Jacobi identity

$$\epsilon(c,a)\langle\langle u, v\rangle, w\rangle + \text{cyclic} = 0 \tag{2}$$

for $u \in \mathfrak{g}_a$, $v \in \mathfrak{g}_b$ and $w \in \mathfrak{g}_c$. Similarly, one may give the definition of a Lie Γ-graded algebra in the opposite direction. That is, one may say that a Lie Γ-graded algebra is a space $\mathfrak{g}$ equipped with a bracket satisfying the two identities (1) and (2). If one chooses this path, then one will quickly see that the function ϵ must necessarily be a commutation factor. However, since in this paper we are primarily aiming at determining a particular classification of commutation factors, we do not wish to spend much time on this question and instead refer the reader to [4].

Notice that for every commutation factor, either $\ker\delta = \Gamma$ or $\ker\delta$ is a subgroup of index two in Γ. We have some terminology to distinguish the types of Lie Γ-graded algebras which result. In the former case, $\mathfrak{g}$ is called a "Lie color algebra" and in the latter case $\mathfrak{g}$ is a "Lie color *super*algebra".

3. The cases $\Gamma = (\mathbb{Z}/2\mathbb{Z})^n$

As the question was posed in [1], throughout this discussion we assume that Γ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ for some n. In particular, we give here a classification of commutation factors for such groups. First of all, when we assume Γ is one of these groups, we have some strong constraints on the image of ϵ.

Proposition 3.1. *Suppose Γ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ for some n and suppose ϵ is a commutation factor. Then* (*a*) *the image of ϵ lies in the two element group $C_2 = \{\pm 1\}$ and* (*b*) $\epsilon(a, b) = \epsilon(b, a)$ *for all $a, b \in \Gamma$.*

Proof. (a) As before, the map $a \mapsto \epsilon(a, g)$, by definition, is a homomorphism, so every element x in the image of ϵ must satisfy $x^2 = 1$. The solution set to this equation is $\{\pm 1\}$. (b) Suppose $\epsilon(a, b) = 1$. Then we must have $\epsilon(b, a) = 1$ because $\epsilon(a, b)\epsilon(b, a) = 1$ by definition. Similarly, if $\epsilon(a, b) = -1$, we must have $\epsilon(b, a) = -1$. □

For each n we have the problem of classifying commutation factors for $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^b$. Obviously, every such commutation factor is determined by the values $\epsilon(a_i, a_j)$ for a basis $\{a_1, \dots, a_n\}$ of Γ, so this helps to formulate a notion of equivalence.

Definition 3.2. Suppose ϵ_1 and ϵ_2 are two commutation factors for $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^n$. Then ϵ_1 and ϵ_2 are equivalent if there are bases $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ for Γ such that $\epsilon_1(a_i, a_j) = \epsilon_2(b_i, b_j)$ for all i and j.

If ϵ is an arbitrary commutation factor for Γ, there may be some "redundancies" which we wish to eliminate. That is, if we think of commutation factors for $(\mathbb{Z}/2\mathbb{Z})^n$ as particular symmetric matrices with entries from $\{\pm 1\}$, then we would like to ignore those with more than one row or column filled with all 1s, as these can be reduced. Elaborating on this, we have the following definition.

Definition 3.3. Given Γ and a commutation factor ϵ, let

$$\Gamma_0 = \{a \in \Gamma : \epsilon(a, b) = 1 \ \text{ for all } \ b \in \Gamma\}.$$

If Γ_0 consists of exactly one element, then ϵ is called a reduced commutation factor.

Evidently Γ_0 is a subgroup of Γ. Accordingly, we can obtain a new commutation factor for the group Γ/Γ_0 by setting

$$\epsilon_0(a + \Gamma_0, b + \Gamma_0) = \epsilon(a, b),$$

and it is also clear that such a commutation factor ϵ_0 is reduced.

It is clear that if ϵ_1 and ϵ_2 are equivalent, then so are their reductions. We say that two pairs (Γ_1, ϵ_1) and (Γ_2, ϵ_2) of groups and corresponding commutation factors are equivalent if the reductions of ϵ_1 and ϵ_2 are equivalent. Let $\mathbb{E}$ denote the set of equivalence classes of these pairs. As we have seen, for each element of $\mathbb{E}$, there is a "representative" reduced commutation factor, and vice versa. For this reason, we may speak of the elements of $\mathbb{E}$ either as equivalence classes of pairs (Γ, ϵ) or simply as reduced commutation factors. Thus our problem of classifying commutation factors reduces to that of classifying reduced commutation factors. To obtain this classification, we need a couple key lemmas.

Lemma 3.4. *Suppose* $\epsilon(a, b) = 1$ *for all* $a, b \in \ker(\delta)$ *with* a *and* b *distinct and non-zero. Then* ϵ *is a non-reduced commutation factor.*

Proof. This is trivial if $\ker \delta = \Gamma$, so we assume there are elements $c \in \Gamma$ with $\delta(c) = -1$. The proof is by contradiction. Suppose $a, b \in \ker \delta$ is any such pair described in this lemma. If ϵ were reduced then we would have $\epsilon(a, c) = \epsilon(b, c) = -1$ for every element $c \in \Gamma$ with $\delta(c) = -1$. That is, the homomorphisms $c \mapsto \epsilon(a, c)$ and $c \mapsto \epsilon(b, c)$ would be non-trivial. Since a and b are distinct and non-zero, we would therefore have $\epsilon(a+b, c) = 1$ for all $c \in \Gamma$. Since $a+b$ is non-zero, ϵ would be non-reduced, leading to a contradiction. $\square$

The contrapositive of this lemma yields the following corollary.

Corollary 3.5. *If* ϵ *is reduced and* $\operatorname{rank}(\ker \delta) \geq 2$, *then there is a pair* $\{a_1, a_2\} \subset \Gamma$ *with* $\delta(a_i) = 1$ *and* $\epsilon(a_1, a_2) = -1$.

This allows us to state the following lemma.

Lemma 3.6. *Choose* $a_1, a_2 \in \Gamma$ *such that* $\delta(a_i) = 1$ *for both* i *and* $\epsilon(a_1, a_2) = -1$ *and suppose* $\{a_1, a_2, b_3, \ldots, b_n\}$ *is a basis for* Γ. *For each* $i = 3, 4, \ldots, n$ *define elements* $a_i \in \Gamma$ *by*

$$a_i = \begin{cases} b_i & \text{if } \epsilon(a_1, b_i) = 1, \ \ \epsilon(a_2, b_i) = 1, \\ b_i + a_1 & \text{if } \epsilon(a_1, b_i) = 1, \ \ \epsilon(a_2, b_i) = -1, \\ b_i + a_2 & \text{if } \epsilon(a_1, b_i) = -1, \ \ \epsilon(a_2, b_i) = 1, \\ b_i + a_1 + a_2 & \text{if } \epsilon(a_1, b_i) = -1, \ \ \epsilon(a_2, b_i) = -1. \end{cases}$$

Then (*a*) *the set* $\{a_1, a_2, a_3, \ldots, a_n\}$ *is a basis for* Γ *and* (*b*) $\epsilon(a_i, a_j) = 1$ *for* $i = 1, 2$ *and* $j = 3, 4, \ldots, n$.

Proof. Part (a) is true because each a_i is obtained by elementary row operations and part (b) is true by construction. $\square$

There is a useful interpretation of this lemma, but for this we first need to define a tensor product of commutation factors, analogous to the tensor products of matrices. Suppose $\Gamma_1 \cong (\mathbb{Z}/2\mathbb{Z})^{n_1}$ and $\Gamma_2 \cong (\mathbb{Z}/2\mathbb{Z})^{n_2}$ and that ϵ_1 and ϵ_2 are accompanying commutation factors. Then the tensor product $\epsilon_1 \otimes \epsilon_2$ is a commutation factor for the direct sum $\Gamma_1 \oplus \Gamma_2$

and is given by the following formula:

$$(\epsilon_1 \otimes \epsilon_2)((a_1, a_2), (b_1, b_2)) = \epsilon_1(a_1, b_1)\epsilon_2(a_2, b_2)$$

for $a_i, b_i \in \Gamma_i$. We sometimes omit the $\otimes$ symbol when writing a tensor product. Also, there is a reduced commutation factor which serves as the identity, namely the identity map itself $I : \mathrm{Id} \times \mathrm{Id} \to \mathrm{Id}$. The tensor product on $\mathbb{E}$, induced in the obvious way from the tensor product between commutation factors, is commutative and associative, and this helps to justify our omission of the symbol $\otimes$ when writing tensor products.

There are a couple of non-trivial reduced commutation factors which are important enough to be named. We call them A and B and they can be given by the following tables:

A	0	1
0	1	1
1	1	−1

B	00	01	10	11
00	1	1	1	1
01	1	1	−1	−1
10	1	−1	1	−1
11	1	−1	−1	1

Evidently we associate A with $\mathbb{Z}/2\mathbb{Z}$ and B with the Klein 4-group $K_4 = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

Now we can interpret the preceding Lemma 3.6 as follows.

Corollary 3.7. *Suppose ϵ is a reduced commutation factor for Γ such that* $\mathrm{rank}(\ker \delta) \geq 2$. *Then there is a reduced commutation factor ϵ' for Γ/K_4 such that ϵ and $\epsilon' \otimes B$ are equivalent.*

Finally, as a corollary to this we obtain the classification of reduced commutation factors.

Theorem 3.8. *Suppose ϵ is a reduced ϵ commutation factor for $\Gamma = (\mathbb{Z}/2\mathbb{Z})^n$. Then* (*a*) *if* $\ker \delta = \Gamma$, *then ϵ is equivalent to $B^{n/2}$,* (*b*) *if* $\ker \delta \cong \Gamma/(\mathbb{Z}/2\mathbb{Z})$ *and n is odd, then ϵ is equivalent to $AB^{(n-1)/2}$, or* (*c*) *if* $\ker \delta \cong \Gamma/(\mathbb{Z}/2\mathbb{Z})$ *and n is even, then ϵ is equivalent to $A^2B^{(n-2)/2}$.*

Proof. Using the tensor-product interpretation of our Lemma 3.6, we may inductively "factor" to obtain $\epsilon = \epsilon' \otimes B^r$, where ϵ' is a reduced commutation factor for a group of rank two or less. The three cases (a), (b), and (c) allow for the three possible structures that ϵ' may have. If $\ker \delta = \Gamma$, ϵ' is equivalent to B, giving us part (a). If n is odd, then, because ϵ is reduced, ϵ' must be a non-trivial commutation factor for the group $\mathbb{Z}/2\mathbb{Z}$, i.e. it must be equivalent to A. This gives us case (b). Finally, if n is even and $\ker \delta$ is a proper subgroup of Γ, implying that ϵ' is not equivalent to B, then ϵ' must be equivalent to A^2, giving us part (c). □

There is a useful and interesting relation involving A and B.

Proposition 3.9. *$A \otimes A \otimes A$ and $A \otimes B$ are equivalent.*

Proof. Suppose $\{a_1, a_2, a_3\}$ is a basis for $(\mathbb{Z}/2\mathbb{Z})^3$ such that $\epsilon(a_i, a_i) = -1$ for all i and

$\epsilon(a_i, a_j) = 1$ for all $i \neq j$. Then, by construction, ϵ is equivalent to $A \otimes A \otimes A$. Consider a new basis $\{b_1, b_2, b_3\}$, where

$$b_1 = a_2 + a_3, \qquad b_2 = a_1 + a_3, \qquad b_3 = a_1 + a_2 + a_3.$$

It is easy to compute that the values $\epsilon(b_i, b_j)$ for this basis are given in the table:

ϵ	b_1	b_2	b_3
b_1	1	−1	1
b_2	−1	1	1
b_3	1	1	−1

This exhibits ϵ as being equivalent to $A \otimes B$. □

Because of this relation, we can restate the classification Theorem 3.8 more concisely.

Theorem 3.10. *Every non-trivial reduced commutation factor is equivalent to either A^n or B^n for some positive integer n.*

4. 2-Manifolds

As it turns out, the classification of reduced commutation factors coincides exactly with that for connected compact 2-manifolds. Let $\mathbb{M}$ denote the set of homeomorphism-equivalence classes of connected compact 2-manifolds. For any two elements X_1 and X_2 in $\mathbb{M}$, we can form the connected sum $X_1 \# X_2$. This is obtained cutting discs out of X_1 and X_2 and then identifying the resulting boundaries homeomorphic to S^1. This operation is commutative and associative and the sphere S^2 serves as an identity element. If $P = \mathbb{R}P^2$ denotes the real projective plane and $T = S^1 \times S^1$ denotes the torus, then we have the following proposition, from [5].

Proposition 4.1. *$P\#P\#P$ and $P\#T$ are homeomorphic.*

We can now define a map φ of $\mathbb{M}$ into $\mathbb{E}$

Proposition 4.2. *There is a bijection $\varphi : \mathbb{M} \to \mathbb{E}$ which is uniquely determined by the conditions:* (*a*) $\varphi(P) = A$, (*b*) $\varphi(T) = B$, *and* (*c*) $\varphi(X\#Y) = \varphi(X) \otimes \varphi(Y)$.

Proof. This follows from the complete classification of closed compact 2-manifolds. See [2]. □

Many readers should recognize that φ merely delivers a kind of *intersection matrix* for these manifolds, and that the preservation of the "algebraic" relation is merely a reflection of the relationship of the connected sum with the intersection matrix. Thus, reduced commutation factors give a faithful invariant for compact connected 2-manifolds.

References

[1] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1) (1977) 8–96.
[2] J.J. Rotman, An Introduction to Algebraic Topology, Springer, New York, 1988.
[3] V. Rittenberg, D. Wyler, Generalized superalgebras, Nucl. Phys. B 139 (3) (1978) 189–202.
[4] M. Scheunet, Generalized Lie algebras, J. Math. Phys. 20 (4) (1979) 712–720.
[5] J. Stillwell, Classical Topology and Combinatorial Group Theory, Springer, New York, 1980.